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Obtuse Cones in Hilbert Spaces and Applications to Partial Differential Equations

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The positive cone K in a partially ordered Hilbert space is said to be obtuse with respect to the inner product if the dual cone $K^* \subset K$. Obtuseness of cones with respect to non-symmetric bilinear forms is also defined and characterized. These results are applied to the generalized Sobolev space associated with an elliptic boundary value problem, in particular to the question of determining the non-negativity of the Green's function. A notion of strict obtuseness is defined, characterized and applied to the question of strict positivity of the Green's function. Applications to positivity preserving semi-groups are also given.

1. INTRODUCTION

In [6] Aronszajn and Smith show that a necessary and sufficient condition for a real reproducing kernel space X to have a non-negative kernel is the following: (A-S) for every $x \in X$ there exists an $\tilde{x}$ such that

$$-\tilde{x} \leqslant x \leqslant \tilde{x} \qquad \text{and} \qquad \|\tilde{x}\|_X \leqslant \|x\|_X . \tag{1.1}$$

This theorem can be reformulated so as to avoid reference to the reproducing kernel structure. Such a formulation, for real function spaces, was given by Deny, [9] and in [8] the theorem was formulated for an arbitrary ordered Hilbert space X. The latter formulation is as follows: let K be the positive cone in X and K^* its dual cone. Then $K^* \subset K$ if and only if (A-S) holds. In regard to the question of non-negativity of Green's functions of elliptic operators, the above results are applicable only to symmetric problems, therefore our first object here has been to obtain a generalization which is applicable to the non-symmetric case. To this end we consider an ordered Hilbert space furnished with a bounded, coercive but not necessarily symmetric bilinear form $\langle\ ,\ \rangle$

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and find necessary and sufficient coditions for the containment of the cones $\{u: \langle , x\rangle \geqslant 0, \text{ all } x \in K\}$, $\{v: \langle x, v\rangle \geqslant 0, \text{ all } x \in K\}$ in K.

For application to the problem of (strict) positivity of the Green's function we next find necessary and sufficient conditions, for example, for $K^* \setminus \{0\}$ to be contained in the quasi-interior (in the sense of Karlin, [16]) of K. One such condition is that $(A\text{-}S)$ holds with the norm inequality in (1.1) replaced by strict inequality for all $x \notin K \cup (-K)$. A criterion more useful for application to differential equations, but valid only for function spaces, is the non-existence of a non-trivial order direct sum decomposition of X. Results of this type are also given for the non-symmetric case.

We next use our generalization of the Aronszajn–Smith theorem to prove the characterization of Deny [9] and Ito [15] of those real functional Hilbert spaces which are Riesz spaces. We conclude with applications to differential equations, in particular to the biharmonic operator, and to the theory of positivity preserving semi-groups on L^2.

2. Preliminaries

We consider a real ordered Hilbert space X with partial order "$\geqslant$" induced by a closed convex cone K, inner product $[\cdot, \cdot]$, and norm $\|\cdot\|$, (our terminology follows that of [18], [20]).

We assume that there is given on X a continuous, not necessarily symmetric, real-valued bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle u, u\rangle \geqslant c \| u \|^2, \qquad u \in X \tag{2.1}$$

for some positive constant c. There will be no loss of generality in assuming, as we shall, that

$$[u, v] = \tfrac{1}{2}(\langle u, v\rangle + \langle v, u\rangle), \qquad u, v \in X, \tag{2.2}$$

so that (2.1) can be replaced by

$$\langle u, u\rangle \geqslant \| u \|^2, \qquad u \in X. \tag{2.3}$$

Our assumptions on $\langle \cdot, \cdot \rangle$ clearly imply the existence of continuous linear operators T and $S = T^{-1}$ on X satisfying

$$\langle u, v\rangle = [u, Tv], \qquad u, v \in X, \tag{2.4}$$

and

$$\langle u, Sv\rangle = [u, v], \qquad u, v \in X.$$

From (2.3) it follows that

$$\| S \| \leqslant 1. \tag{2.6}$$

We define the cones K^*, K_a^* and ${}_aK^*$ as follows:

$$K^* = \{u \in X: [v, u] \geqslant 0 \text{ for all } v \in K\},$$
$$K_a^* = \{u \in X: \langle u, v\rangle \geqslant 0 \text{ for all } v \in K\},$$
$${}_aK^* = \{u \in X: \langle v, u\rangle \geqslant 0 \text{ for all } v \in K\}.$$

We note that

$$K_a^* = S^*K^*, \qquad K^* = T^*K_a^* \tag{2.7}$$

$${}_aK^* = SK^*, \qquad K^* = T_aK^* \tag{2.8}$$

A continuous linear functional l on X is a *positive linear functional* if l is not the zero functional and $l(u) \geqslant 0$ for every $u \in K$. If l has one of the representations: $l = [u, \cdot] = [\cdot, u]$, $u \in K^*\backslash\{0\}$; $l = \langle \cdot, w\rangle$, $w \in K_a\backslash\{0\}$; $l = \langle v, \cdot\rangle$, $v \in {}_aK_a^*\backslash\{0\}$ then l is a positive linear functional. Conversely, a positive linear functional can be represented in each of these three ways.

Two elements $x, y \in K$ will be called *disjoint* if

$$\{u: 0 \leqslant u \leqslant x\} \cap \{v: 0 \leqslant v \leqslant y\} = 0;$$

in particular, if X is a real function space with the natural order then two non-negative functions are disjoint if they have disjoint support, and if X is a Riesz space, i.e., a vector lattice with respect to $\geqslant$, then two elements in K are disjoint in the above sense if and only if they are lattice disjoint.

The form $\langle \cdot, \cdot\rangle$ will be said to be *local* if $\langle u, v\rangle = 0$ whenever $u, v \in K$ are disjoint and *semi-local* if $\langle u, v\rangle \leqslant 0$ whenever $u, v \in K$ are disjoint.

We remark that if X is a Riesz space with respect to $\geqslant$ then X is a *Hilbert lattice* in the sense of [20] if and only $[\cdot, \cdot]$ is local and $\| x \| \geqslant \| y \|$ whenever $x \geqslant y \geqslant 0$. Our interest here however is in the case in which X is a Riesz space but not a Hilbert lattice.

We conclude this section with a brief discussion of the finite dimensional case. Let $A = (a_{ij})$ be a real $n \times n$ matrix such that $B = \frac{1}{2}(A + {}^tA) = (b_{ij})$ is positive definite, let X be R^n furnished with the inner product $[x, y] = (x, By) = \sum_{i=1}^n \sum_{j=1}^n b_{ij} x_i y_j$, and partially ordered so that $x \geqslant y$ if and only if $x_i \geqslant y_i$ for $i = 1,..., n$. Finally, let $\langle x, y\rangle = (x, Ay) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i y_j$. Then X is a lattice and it follows from Theorem 2.4, [17] that any finite-dimensional

Hilbert space with a lattice structure can be represented in this way. The cone K is given by

$$K = \{x = (x_1, \dots, x_n) : x_i \geqslant 0, i = 1, \dots, n\}$$

and

$$K^* = B^{-1}K, \qquad {}_aK^* = A^{-1}K, \qquad K_a^* = ({}^tA)^{-1}K.$$

The form $\langle \cdot, \cdot \rangle$ is local if and only if A is diagonal and is semi-local if and only if the off-diagonal elements of A are non-positive, i.e., if and only if A is an M-matrix.

3. Obtuse Cones

We shall say that the cone K is *obtuse* with respect to the form $\langle \cdot, \cdot \rangle$ if $K_a^* \subset K$. In this section we will develop criteria for the obtuseness of the cone K. These results may be regarded as a generalization of the main theorem of Aronszajn–Smith [6]; see Remark 3.2 below.

Some preliminaries to the main result of this section are the following.

Lemma 3.1. *Let $x \in X$ and let u be the nearest element to x in K, then $v = u - x \in K^*$, and $Sv \in {}_aK^*$ and $[u, v] = 0$.*

Proof. For all $\xi \in K$ we have

$$\| \xi - x \|^2 \geqslant \| u - x \|^2 = \| v \|^2$$

or

$$[\xi - u, v] \geqslant 0. \tag{3.1}$$

Upon taking $\xi = 0$ and $\xi = 2u$ in (3.1) we see that

$$[u, v] = 0 \tag{3.2}$$

so that, from (3.1)

$$[\xi, v] \geqslant 0 \qquad \text{for all } \xi \in K$$

and thus $v \in K^*$, $Sv \in {}_aK^*$.

Corollary 3.1. *The cone K^* is total for X, i.e. $[x, v] = 0$ for all $v \in K^*$ implies $x = 0$.*

Proof. If $x \notin K$, let u, v be as in Lemma 3.1. Then $0 \neq v \in K^*$ and, by (3.2), $[x, v] = -[u - x, v] = -\| v \|^2 \neq 0$. If $x \in K$ apply the same argument to $-x$.

THEOREM 3.1. *The following are equivalent*

(a) $K_a^* \subset K$.

(b) ${}_aK^* \subset K$.

(c) *Every* $x \in X$ *can be represented in the form* $x = u - v$ *with* $[u, v] = 0$, $u \in K$, $Sv \in K$.

(d) *Every* $y \in X$ *can be represented in the form* $y = u - w$ *with* $\langle u, w \rangle = 0$, $u \in K$, $Su - y \in K$.

(e) *Every* $x \in X$ *can be represented in the form* $x = u - v$ *with* $[u, v] \leqslant 0$, $u \in K$, $Sv \in K$.

(f) *For every* $x \in X$ *there exists an* $\tilde{x} \in X$ *with* $\| \tilde{x} \| \leqslant \| x \|$, $\tilde{x} + x \in K$ *and* $S(\tilde{x} - x) \in K$.

(g) *For every* $x \notin K$ *there exists a* $w \in K$ *such that* $\langle x, w \rangle < 0$.

Proof. (a) implies (b). Let $u \in {}_aK^*$. If $u \notin K$ then there is a continuous linear functional l such that $l(v) = 0$ for $v \in K$ and $l(u) < 0$. Since l is positive it can be represented as $l = \langle w, \cdot \rangle$, $w \in K_a^* \subset K$ and hence $l(u) = \langle w, u \rangle \geqslant 0$, which is a contradiction.

(b) implies (c). Let $x \in X$ then by Lemma 3.1, $x = u - v$ where $[u, v] = 0$, $u \in K$, $Sv \in {}_aK^*$, hence $Sv \in K$, by (b).

(c) is equivalent to (d). Let x and y be related by $x = Ty$, $y = Sx$, and with u and v as in (c) let $w = u - y$. Then $Su - y = S(u - x) = Sv$ and $\langle u, w \rangle = \langle u, u - y \rangle = \langle u, u \rangle - \langle u, Sx \rangle = [u, u] - [u, x] = [u, v]$. This establishes the equivalence.

(c) implies (e). This implication is trivial.

(e) implies (f). Let $x \in X$ and let u, v be as in (e). Put $\tilde{x} = 2u - x$, then $\tilde{x} + x = 2u \in K$ and $\tilde{x} - x = 2(u - x) = 2v$ so that $S(\tilde{x} - x) = 2Sv \in K$. Finally, $\| \tilde{x} \|^2 - \| x \|^2 = [\tilde{x} - x, \tilde{x} + x] = 4[u, v] \leqslant 0$ so that $\| \tilde{x} \|^2 \leqslant \| x \|^2$.

(f) implies (g). Let $x \notin K$, $\tilde{x}$ as in (f). Then $\tilde{x} \neq x$ since $\tilde{x} + x \in K$ by (f). Let $w = S(\tilde{x} - x)$. Then $\langle x, w \rangle = [x, \tilde{x} - x] = \frac{1}{2}[x - \tilde{x}, \tilde{x} - x] + \frac{1}{2}[x + \tilde{x}, \tilde{x} - x] = -\frac{1}{2} \| x - \tilde{x} \|^2 + \frac{1}{2}[\| \tilde{x} \|^2 - \| x \|^2] < 0$, since $\| \tilde{x} \|^2 \leqslant \| x \|^2$ by (f), and $\| x - \tilde{x} \|^2 > 0$.

(g) implies (a). This implication is immediate.

In the case where $\langle \cdot, \cdot \rangle = [\cdot, \cdot]$ we have $K_a^* = {}_aK^* = K^*$ and $S = I =$ the identity. Thus we immediately obtain the following result.

COROLLARY 3.1. *The following are equivalent.*

(a) $K^* \subset K$.

(b) *For every* $x \in X$, $x = u - v$ *where* $u \in K$, $v \in K^* \subset K$, $[u, v] = 0$,

(c) *For every* $x \in X$, $x = u - v$ *where* $u \in K$, $v \in K^* \subset K$, $[u, v] \leqslant 0$,

(d) *For every* $x \in X$ *there is an* $\tilde{x} \in K$ *such that* $-\tilde{x} \leqslant x \leqslant \tilde{x}$ *and* $\| \tilde{x} \| \leqslant \| x \|$.

COROLLARY 3.2. *The following are equivalent.*

(a) $K_a^* = K$.

(b) ${}_aK^* = K$.

(c) $K_a^* \subseteq K$ *and*

$$\langle u, v \rangle \geqslant 0 \qquad \textit{for all } u, v \in K. \tag{3.3}$$

Proof. Clearly (3.3) implies both $K^* \supset K$ and ${}_aK^* \supset K$, so that, in view of Theorem 3.1 both (a) and (b) follow from (c). Conversely, either (a) or (b) implies (3.1) and the inclusion $K^* \subseteq K$.

COROLLARY 3.3. *If any of the conditions* (a), (b) *or* (c) *of Corollary* 3.2 *hold then*

$$[u, v] \geqslant 0 \qquad \textit{for all } u, v \in K,$$

and

$$\| u + v \| \geqslant \| v \| \qquad \textit{for all } u, v \in K,$$

furthermore, $K \subseteq K^*$.

COROLLARY 3.4. *Let* X *be a Riesz space with respect to* $\geqslant$ *and let* $\langle \cdot, \cdot \rangle$ *hence also* $[\cdot, \cdot]$ *be semi-local. Then* K_a^*, ${}_aK^*$, $K^* \subset K$.

Proof. For $x \in X \backslash K$ we have $x = x_+ - x_-$ where $x_+, x_- \in K$, $x_+ \wedge x_- = 0$. Moreover, $\langle x, x_- \rangle = \langle x_+, x_- \rangle \| x_- \|^2 \leqslant - \| x_- \|^2 < 0$. Therefore condition (g) of Theorem 3.1 holds and it follows that K_a^*, ${}_aK^* \subset K$. The same argument can be applied with the form $\langle \cdot, \cdot \rangle$ replaced by $[\cdot, \cdot]$ to show $K^* \subset K$.

COROLLARY 3.5. *Let* X *be as in Corollary* 3.4. *If any of the equivalent con-conditions of Corollary* 3.2 *hold then* X *is a Hilbert* lattice *and* $\langle \cdot, \cdot \rangle$ *is local.* X *is a Hilbert lattice if and only if* $K = K^*$.

Remark 3.1. It is possible to have K_a^*, ${}_aK^* \subset K$ but $K^* \not\subset K$. For example, let X be R^3, $K = \{(x, x_2, x_3): x_i \geqslant 0, i = 1, 2, 3\}$, and $\langle u, v \rangle = (Au, v)$ where

$$A = \begin{bmatrix} 1 & -1 & \frac{1}{4} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then A is positive definite and has a positive inverse so that $K_a^* \subset K$. However $A + {}^tA$ is a positive definite matrix whose inverse is not a positive matrix and thus $K^* \not\subset K$. This example shows also that, when $K_a^* \subset K$ it is not in general possible to write $x = u - v$ with $u, v \in K$ and both $\langle u, v \rangle$, $\langle v, u \rangle \leqslant 0$ for $x \in X$.

Remark 3.2. The equivalence of conditions (a) and (d) of Corollary 3.1 is equivalent to the real case of the main result of Aronszajn–Smith [6]. Indeed, by Corollary 3.1, K^* is total for X, thus X can be identified with a space of functions on K^* by the rule $x \to x(\cdot)$ where $x(y) = [x, y]$, $y \in K^*$, clearly elements of K are precisely those that correspond, by this rule, to non-negative functions. With this identification, X becomes a reproducing kernel space with reproducing kernel $k_{(\cdot)}$ satisfying

$$x(y) = [x, k_y], \qquad y \in K^*, \quad x \in X,$$

or

$$k_y = y, \qquad y \in K^*.$$

Clearly k_y is non-negative as a function, i.e., $k_y(y') = [y', y] \geqslant 0$ for all $y' \in K^*$ if and only if $K^* \subset K$.

4. Striclty Obtuse Cones

We shall here say that $x \in K$ is a *quasi-interior element* of K if $l(x) > 0$ for every positive linear functional l on X. The set of quasi-interior elements of K will be denoted by Q. We shall say that K is *strictly obtuse* with respect to the form $\langle \cdot, \cdot \rangle$ if $K_a^* \backslash \{0\} \subset Q$, and this section will be concerned with criteria for K to be strictly obtuse.

We shall require the following characterization of quasi-interior elements.

Lemma 4.1. *The following are equivalent*:

(a) *u is a quasi-interior element of K,*

(b) *$u \in K$ and the set $C_u = \{\alpha u - \xi, \alpha > 0, \xi \in K\}$ is dense in X.*

Proof. (b) implies (a). Suppose that $u \in K$ and C_u is dense in X and let $v \in K^*$. If $x = \alpha u - \xi$ where $\alpha > 0$ and $\xi \in K$ then

$$\begin{aligned}[v, x] &= \alpha[v, u] - [v, \xi] \\ &\leqslant \alpha[v, u].\end{aligned}$$

Consequently, if $[v, u] = 0$ then $[v, x] \leqslant 0$ for all $x \in C_u$ and hence $v = 0$. Since any positive linear functional l has the representation $l = [v, \cdot]$, $v \in K^* \backslash \{0\}$, (a) follows.

(a) implies (b). Let $u \in Q$. The set C_u and hence also the set $\bar{C}_u$ is convex. Let $x \in X$ and let w be the nearest element to x in $\bar{C}_u$. Then

$$[y - w, w - x] \geqslant 0, \qquad \text{all } y \in \bar{C}_u. \tag{4.1}$$

Clearly $\bar{C}_u - K \subseteq \bar{C}_u$ so we can take $y = w - \xi$ in (4.1) for $\xi \in K$ to obtain

$$[\xi, x - w] \geqslant 0 \qquad \text{all } \xi \in K,$$

which implies $x - w \in K^*$. Taking $y = \alpha u$, $\alpha > 0$ in (4.1) we get

$$\alpha[u, x - w] - [w, x - w] \leqslant 0, \qquad \text{all } \alpha \geqslant 0,$$

which implies that $[u, x - w] \leqslant 0$, and hence since $x - w \in K^*$, $u \in Q$, $x = w \in \bar{C}_u$. Since $x \in X$ was arbitrary it follows that C_u is dense in X.

COROLLARY 4.1. *Let X be a Riesz space with respect to $\geqslant$, and suppose that the lattice operations in X are continuous from the strong to the weak topology. Then $u \in Q$ if and only if $u \in K$ and the set $\{x\colon -u \leqslant x \leqslant u\}$ spans a dense subset of X.*

Proof. Since $\{x\colon -u \leqslant x \leqslant u\} \subset C_u$ the condition is clearly sufficient. Conversely, if $u \in Q$ and $v \in K$ then $v = \lim_{n\to\infty} w_n$, $w_n \in C_u$, $n = 1, 2,\ldots,$ thus, in view of the assumed continuity of the lattice operations, $v = wk\text{-}\lim_{n\to\infty} (w_n)_+$. However $(w_n)_+ \in \operatorname{span}\{x\colon -u \leqslant x \leqslant u\}$, consequently the closure of the latter set contains K and hence also X.

Remark 4.1. Corollary 4.1 shows that if X is a Riesz space with weakly continuous lattice operations then an element is quasi-interior to K in the sense of the above definition if and only if it is quasi-interior to K in the sense of Schaefer [20]. For Hilbert lattices this result is proved in [20]. The continuity assumed of the lattice operations is easily verified for the function spaces with which we deal in the remainder of this paper, and also is satisfied whenever the form $[\cdot, \cdot]$ is semi-local.

THEOREM 4.1. *The following are equivalent*:

(a) ${}_aK^* \backslash \{0\} \subset Q$,

(b) $K_a^* \backslash \{0\} \subset Q$,

(c) *For $x \notin K \cup (-K)$ $x = u - v$ with $u \in K$, $Sv \in K$ and $[u, v] < 0$.*

(d) *For $x \notin K \cup (-K)$ there exists an $\tilde{x} \in X$ such that $\| \tilde{x} \| < \| x \|$ and $\tilde{x} + x \in K$, $S(\tilde{x} - x) \in K$.*

Proof. (a) *is equivalent to* (b). From the remark in Section 2 concerning the representation of positive linear functionals it is clear that (a) and (b) are both equivalent to the assertion $\langle u, v\rangle > 0$ whenever $u \in K_a^*\backslash\{0\}$ and $v \in {}_aK^*\backslash\{0\}$.

(c) *is equivalent to* (d). If (c) holds then (d) holds with $\tilde{x} = 2u - x$. Conversely, if (d) holds then (c) holds with $u = \frac{1}{2}(\tilde{x} + x)$, $v = \frac{1}{2}(\tilde{x} - x)$

(c) implies (a). Assume that (c) holds and (a) does not. Then there exists $u \in K_a^*\backslash\{0\}$ and $v \in {}_aK^*\backslash\{0\}$ such that $\langle u, v\rangle = [u, Tv] = 0$. We can suppose further that $\| u \| = \| Tv \| = 1$. Let $x = u - Tv$ and note that $\langle x, v\rangle = -\| Tv \|^2$ so that $x \notin K$. Also $\langle u, -x\rangle = \langle u, Tv - u\rangle < \| u \| (\| Tv \| - \| u \|) = 0$ so that $x \notin -K$. Suppose $\operatorname{span}\{u, Tv\} \neq X$. Then let z be such that $\| z \| = 1$, $[z, u] = [z, Tv] = 0$, and such that in span $\{u, Tv, z\}$, x has the decomposition $x = \xi_1 - \xi_1$ with $\xi_1 \in K$, $S\xi_2 \in K$ and $[\xi_1, \xi_2] < 0$. This is possible since (c) is assumed and $x \notin K \cup (-K)$. We can then write $\xi_i = \alpha_i z + \beta_i u + \gamma_i Tv$, $\alpha_i, \beta_i\gamma_i \in R$, $i = 1, 2$. Moreover

$$\beta_2 = [\xi_2, u] = \langle u, S\xi_2\rangle \geqslant 0, \qquad \text{since} \quad u \in K_a^*, \quad S\xi_2 \in K,$$

and

$$\gamma_1 = [\xi_1, Tv] = \langle \xi_1, v\rangle \geqslant 0, \qquad \text{since} \quad v \in {}_aK^*, \quad \xi_1 \in K.$$

Since $\xi_1 - \xi_2 = u - Tv$ we have

$$\begin{aligned} \alpha_1 &= \alpha_2 \\ \beta_1 &= \beta_2 + 1 \geqslant 0 \\ \gamma_2 &= \gamma_1 + 1 \geqslant 0. \end{aligned}$$

Therefore, $[\xi_1, \xi_2] = \alpha_1^2 + \beta_1\beta_2 + \gamma_1\gamma_2 \geqslant 0$ which is a contradiction. If $\operatorname{span}\{u, Tv\} = X$, omit z from the above argument.

(a) implies (c). Suppose that (a) holds and $x \notin K \cup (-K)$. Since (a) implies condition (c) of Theorem 3.1, $x = u' - v'$ where $u' \in K\backslash\{0\}$ since $x \notin -K$; $v \in {}_aK^*\backslash\{0\}$ since $x \notin K$, and $[u', v'] = 0$. We thus have $Sv' \in Q$ so by Lemma 4.1, given $\epsilon > 0$ there exists $\alpha \geqslant 0$ and $k' \in K$ such that

$$\| Su' - \alpha(Sv' - k')\| < \epsilon \| T \|^{-1}$$

By Theorem 3.1, (d), $y = Su' - \alpha(Sv' - k') = k_1 - k_2$ with $k_1 \in K$, $Sk_z - y \in K$ and $\langle k_1, k_2\rangle = 0$. Thus

$$Su' - \alpha(Sv' - k) = Sk_1 \tag{4.2}$$

where

$$k = k' + \alpha^{-1}(Sk_1 - y) \in K,$$

and upon applying T to both sides of (4.2) we get

$$u' - \alpha(v' - Tk) = k_1 \in K. \tag{4.3}$$

Let $\eta = v' - Tk$. Then

$$\begin{aligned} \| u' - \alpha\eta \|^2 &= \| k_1 \|^2 \\ &= \langle k_1 , k_1 - k_2 \rangle \\ &= [k_1 , T_y] \leqslant \| k_1 \| \epsilon, \end{aligned}$$

or

$$\| u' - \alpha\eta \| < \epsilon,$$

Moreover, by (4.3)

$$u' \geqslant \alpha\eta$$

For $0 < \delta < \min\{1, \alpha\}$ to be determined let

$$u = u' - \delta\eta \in K,$$

so that if $v = u - x$ then

$$Sv = (1 - \delta)\, Sv' + k \in K.$$

Since $x \notin K$, $2u' - x \neq 0$, thus we can choose $\epsilon > 0$ such that

$$\epsilon < \| u' \|^2(\| 2u' - x \|)^{-1},$$

and then

$$\begin{aligned} [\eta, 2u' - x]\ \alpha^{-1}&[\alpha\eta, 2u' - x] \\ &= a^{-1} \| u' \|^2 + \alpha^{-1}[\alpha\eta - u', 2u' - x] \\ &\geqslant \alpha^{-0} \| u' \|^2 - \epsilon\alpha^{-1} \| 2u' - x \| > 0. \end{aligned}$$

Thus if δ is chosen so that

$$\delta < [\eta, 2u' - x] \| \eta \|^{-2}$$

then

$$[u, v] = -\delta[\eta, 2u' - x] + \delta^2 \| \eta \|^2 < 0.$$

Thus (a) implies (c).

COROLLARY 4.2. *The following are equivalent*:

(a) $K^*\backslash\{0\} \subset Q$,

(b) *For* $x \notin K \cup (-K)$, $x = u - v$ *with* $u \in K$, $v \in K$ *and* $[u, v] < 0$,

(c) *For every* $x \notin K \cup (-K)$ *there exists an* $\tilde{x} \in K$ *such that*

$$-\tilde{x} \leqslant x \leqslant \tilde{x} \qquad \textit{and} \qquad \| \tilde{x} \| < \| x \|.$$

5. Functions Spaces with Strictly Obtuse Cones

We shall now assume that X is a real function space and obtain a criterion foı strict obtuseness of the positive cone for this more special case. Let (Ω, Σ) be a measure space and μ a measure on this space. In what follows we shall use the notation

$$(f, g) = \int fg\, d\mu, \qquad | f | = \left(\int | f |^2\, d\mu\right)^{1/2}$$

L^2 will denote the space $L^2(\Omega, \Sigma, \mu)$ of real-valued μ-square integrable functions on Ω. The elements of the Hilbert space X will be assumed to be (equivalence classes of) functions in L^2 and X will be given the natural order, i.e. K consists of those functions $u \in X$ such that $u(x) \geqslant 0$ a.e. on Ω. For a function f in X or L^2 we shall write $f > 0$ if $f(x) \geqslant 0$ a.e. on Ω and $f \neq 0$. We shall say that $u \in X$ is *strictly positive if* $u \in Q$, the quasi-interior of K, and that $f \in L^2$ is *strictly positive* if $f(x) > 0$ a.e. on Ω. We make the following two additional assumptions concerning X: (1) the natural injection i: $X \to L^2$ is continuous, (2) for every $A \in \Sigma$ with $0 < \mu(A) < \infty$ there exists a $u \in X$ such that

$$\int_A u\, d\mu > 0. \tag{5.1}$$

For $\lambda \geqslant 0$, X_λ will denote the Hilbert space that results from furnishing X with the equivalent inner product

$$[u, v]_\lambda = [u, v] + \lambda(iu, iv), \qquad u, v \in X$$

and the corresponding norm

$$\| u \|_\lambda = [u, u]_\lambda^{1/2};$$

also we denote

$$\langle u, v\rangle_\lambda = \langle u, v\rangle + \lambda(iu, iv).$$

The operators T_λ, S_λ, T_λ^*, S_λ^* and the cones K_λ^*, ${}_aK_\lambda^*$, $K_{a,\lambda}^*$ are defined for the ordered spaces X_λ as in Section 2. The results of Sections 3, 4 obviously remain valid for X, when T_λ, S_λ, etc. are substituted for T, S,.... The injection $X_\lambda \to L^2$ will also be denoted by i, its adjoint with respect to $[\cdot, \cdot]_\lambda$ will be denoted by i_λ^*.

The assumptions above concerning X together with Lemma 4.1 immediately imply the following.

Lemma 5.1. *If $u \in X$ is strictly positive (i.e., $u \in Q$) then $iu \in$ is strictly positive.*

Remark. If X is a reproducing kernel space and if there is no point of Ω at which all functions of X vanish then a strictly positive function in X is positive everywhere on Ω.

We shall put

$$\gamma_\lambda = iS_\lambda i_\lambda^* \tag{5.2}$$

and note that if $u = S_\lambda i_\lambda^* f$, $f \in L^2$ then

$$\langle v, S_\lambda i_\lambda^* f\rangle + \lambda(iv, \gamma_\lambda f) = (iv, f), \qquad \text{all} \quad v \in X_\lambda . \tag{5.3}$$

Upon putting $v = S_\lambda i_\lambda^* f$ we get

$$\| S_\lambda i_\lambda^* f \|^2 + \lambda \,|\, \gamma_\lambda f \,|^2 = (\gamma_\lambda f, f)$$

from which follows

$$|\, \gamma_\lambda f \,| \leqslant (\| i \|^{-2} + \lambda)^{-1} \,|\, f \,|. \tag{5.4}$$

LEMMA 5.2 *Let* $\lambda, \lambda' \geqslant 0$ *and let*

$$|\, \lambda' - \lambda \,| < \| i \|^{-2} + \lambda \tag{5.5}$$

then

$$\gamma_{\lambda'} = \sum_{n=1}^{\infty} (\lambda - \lambda')^{n-1} \gamma_\lambda ; \tag{5.6}$$

the series on the right in (5.6) *converges in the operator norm on* L^2.

Proof. The convergence assertion is clear and, in view of the Neumann expansion, (5.6) is equivalent to the relation

$$\gamma_{\lambda'} = \gamma_\lambda (I - (\lambda - \lambda')\, \gamma_\lambda)^{-1}. \tag{5.7}$$

To verify (5.7) we rewrite (5.3)

$$\begin{aligned} \langle v, S_\lambda i_\lambda^* f\rangle_{\lambda'} &= \langle v, S_\lambda i_\lambda^* f\rangle + \lambda'(iv, i(S_\lambda i_\lambda^* f)) \\ &= (iv, f - (\lambda - \lambda')\, \gamma_\lambda f), \qquad \text{for all} \quad v \in X, \end{aligned}$$

from which it follows that

$$S_\lambda i_\lambda^* f = S_{\lambda'} i_{\lambda'}^* (f - (\lambda - \lambda')\, \gamma_\lambda f)$$

hence

$$\gamma_\lambda f = \gamma_{\lambda'} (f - (\lambda - \lambda')\, \gamma_\lambda f),$$

and since $f \in L^2$ is arbitrary this is equivalent to (5.7).

We say that γ_λ is *positive* if $f > 0$ implies $\gamma_\lambda f > 0$ and *strictly positive* if $f > 0$ implies $\gamma_\lambda f$ is strictly positive.

LEMMA 5.3. *Let $\lambda \geqslant 0$, then the following are equivalent*:

(a) ${}_aK_\lambda^* \subset K$.

(b) γ_λ *is positive.*

Proof. (a) implies (b). From the relation

$$\langle u, S_\lambda i_\lambda^* f\rangle_\lambda = (iu, f)$$

it follows that $S_\lambda i_\lambda^* f \in {}_aK_\lambda^*$ when $f \geqslant 0$. It follows from Theorem 3.1 and assumption 2) of the first paragraph of this section that $S_\lambda i_\lambda^* f \neq 0$ if $f \neq 0$ and thus γ_λ is positive if ${}_aK_\lambda^* \subset K$.

(b) implies (a). Let $u \in K_{a,\lambda}^*$ and let $g \in L^2$ with $g > 0$. Then the positivity of γ_λ implies that $S_\lambda i_\lambda^* g \in K$ and thus

$$(iu, g) = \langle u, S_\lambda i_\lambda^* g\rangle_\lambda \geqslant 0,$$

since g was arbitrary it follows that $u \in K$. Thus $K_{a,\lambda}^* \subset K$, and it follows from Corollary 3.2 that ${}_aK_\lambda^* \subset K$.

Lemma 5.2 immediately implies the following.

LEMMA 5.4. *Let γ_λ' be positive (equivalently let ${}_aK_\lambda^{*\prime} \subset K$) for some $\lambda' > 0$ then γ_λ is positive and ${}_aK_\lambda^* \subset K$ for $0 \leqslant \lambda < \lambda'$.*

We shall say that γ_λ is *ergodic* if for every $f, g \in L^2$ with $f, g > 0$ there exists a positive integer n such that $(\gamma_\lambda{}^n f, g) > 0$. It follows from Lemma 5.2 that if $\gamma_{\lambda'}$ is positive and ergodic then γ_λ is strictly positive for $0 \leqslant \lambda < \lambda'$. The converse also follows immediately from the Neumann expansion. We have in fact the following stronger result

LEMMA 5.5. *Let $\lambda' > 0$. The following are equivalent*:

(a) $\gamma_{\lambda'}$ *is ergodic and positive,*

(b) ${}_aK_\lambda^* \subset Q \cup \{0\}$ *for* $0 \leqslant \lambda < \lambda'$.

Proof. (b) implies (a). As was shown in the proof of Lemma 5.3, $f > 0$ implies $S_\lambda^* i_\lambda^* f \subset {}_aK_\lambda^* \backslash \{0\}$, and therefore (b) implies that γ_λ is strictly positive for all λ: $0 \leqslant \lambda < \lambda'$ hence γ_λ' is ergodic.

(a) implies (b), If γ_λ' is ergodic then for $f > 0$, $0 \leqslant \mu < \lambda'$, $S_\mu i_\mu^* f \in K$ and $\gamma_\mu f$ is strictly positive. Let $0 \leqslant \lambda < \mu < \lambda'$, we have

$$\langle u, S_\mu i_\mu^* f\rangle_\lambda + (\mu - \lambda)(iu\ \gamma_\mu^* f) = \langle u, S_\mu i_\mu^* f\rangle_\mu = (f, iu),$$

and thus $u \in K_{a,\lambda}^*$ implies, since ${}_aK_\lambda^* \subset K$, that $(f, iu) > 0$ whenever $f > 0$, thus u is strictly positive. Now let $v \in {}_aK_\mu^* \backslash \{0\} \subset K$, then, since $u \in K_{a,\lambda}^*$ and is strictly positive

$$\langle u, v\rangle_\mu = \langle u, v\rangle_\lambda + (\mu - \lambda)(iv, iu) > 0$$

and thus, as $v \in {}_aK_\mu^*$ was arbitrary, $u \in \underset{\sim}{Q}$. We have therefore $K_{a,\lambda}^* \subset \underset{\sim}{Q} \cup \{0\}$ and thus by Theorem 4.1 ${}_aK_\lambda^* \subset \underset{\sim}{Q} \cup \{0\}$. This completes the proof.

DEFINITION 5.1. A projection P in X or L^2 will be called an *order projection* if Pu and $(I - P)\,u$ are lattice disjoint in L^2, i.e. if P is multiplication by the characteristic function of a set $A \in \Sigma$. The order projection P is *proper* if $P \neq I$ and $P \neq 0$. If P is a proper order projection then PL^2 is a *proper order direct* summand of L^2. Finally, X will be called *indecomposable* if it admits no proper orthogonal order projection.

LEMMA 5.6 (Ando [5]). *Let γ_λ be positive, then γ_λ is ergodic if and only if it admits no proper direct summand of L^2 as an invariant manifold.*

Proof. Suppose $\gamma_\lambda P = P\gamma_\lambda P$ where P is a proper order projection, then there exist $f, g \in L^2$ with $f, g > 0$, $Pf = f$, $Pg = 0$ and we then have $(\gamma_\lambda{}^n f, g) = 0$ for all n and γ_λ is not ergodic.

Conversely, let $f, g \in L^2$ with $f, g > 0$ and suppose that $(\gamma^n f, g) = 0$, $n = 1, 2, \ldots$. Let $A = \{x : \gamma_\lambda{}^n f(x) = 0, n = 1, \ldots\}$, and let $Ph = \chi_A h$, $h \in L^2$, where χ_A is the characteristic function of A. P is proper since $P\gamma_\lambda{}^n f = \gamma_\lambda{}^n f$ for $n = 1, \ldots$, and $Pg = 0$. Clearly $(I - P) L^2$ is invariant for γ^* and thus PL^2 is invariant for γ_λ.

LEMMA 5.7. *Let $\langle \cdot, \cdot \rangle$ be either symmetric or local and let $\lambda \geqslant 0$, then the following are equivalent:*

(a) *the operator γ_λ admits no proper order direct summand of L^2 as an invariant subspace,*

(b) *X_λ is indecomposable.*

If condition (a) *holds for some $\lambda \geqslant 0$ then it holds for all $\lambda \geqslant 0$.*

Remark. If $\langle \cdot, \cdot \rangle$ is local and X admits any order projection P then P is orthogonal.

Proof. Suppose that X admits the proper orthogonal order projection P and let $\hat{P}$ denote the corresponding order projection on L^2, i.e. $\hat{P}$ is the unique order projection on L^2 such that $iP = \hat{P}i$. For $u \in X$, $f \in L^2$ we have

$$\begin{aligned}
\langle u, S_\lambda i_\lambda^* \hat{P} f \rangle_\lambda &= (iu, \hat{P} f) \\
&= (iPu, \hat{P} f) \\
&= \langle Pu, S_\lambda i_\lambda^* \hat{P} f \rangle_\lambda \\
&= \langle u, P S_\lambda i_\lambda^* \hat{P} f \rangle_\lambda ;
\end{aligned}$$

note that the last step of this computation makes use of the fact that $\langle\cdot,\cdot\rangle$ is either symmetric or local. Since u was arbitrary it follows that $PS_\lambda i^*\hat{P} = S_\lambda i^*\hat{P}$, hence $\hat{P}\gamma_\lambda\hat{P} = \hat{P}iS_\lambda^* i_\lambda^*\hat{P} = iPS_\lambda^* i^*\hat{P} = \gamma_\lambda\hat{P}$. $\hat{P}$ is obviously proper so it follows that (a) implies (b).

Conversely, suppose that γ_λ is positive and $\hat{P}$ is a proper order projection on L^2 such that $\hat{P}\gamma_\lambda\hat{P} = \gamma_\lambda\hat{P}$ and hence also $\hat{P}'\gamma_\lambda\hat{P}' = \gamma_\lambda^*\hat{P}'$ where $\hat{P}' = I - \hat{P}$. Let $M = S_\lambda i^*\hat{P}L^2$ and $N = S_\lambda^* i_\lambda^*\hat{P}'L^2$. Assumption (2) of the first paragraph of this section implies that neither M nor M is $\{0\}$, and since u and v are lattice disjoint in L^2 when $u \in M$ and $v \in N$ it follows that M and N are orthogonal in X when $\langle\cdot,\cdot\rangle$ hence also $[\cdot,\cdot]$ is local. In the symmetric case we have $S_\lambda = S_\lambda^* =$ identity and thus

$$[i_\lambda^*, \hat{P}f, i_\lambda^*\hat{P}'g]_\lambda = (\hat{P}f, \gamma_\lambda\hat{P}'g)$$

and the same is true. Let $\overline{M}$, $\overline{N}$ be the closures of M and N in X_λ. Then $\overline{M}$ and $\overline{N}$ are orthogonal and $\hat{P}i\overline{M} = i\overline{M}$, $\hat{P}'i\overline{N}$. It remains to show that $X_\lambda = \overline{M} \oplus \overline{N}$. This is clear in the symmetric case since the range of i^* is dense. Consider the non-symmetric case and let $w \in X$ be orthogonal to both $\overline{M}$ and $\overline{N}$. Then for all $f \in L^2$ we have

$$0 = [w, S_\lambda i_\lambda^*\hat{P}f]_\lambda = (iS_\lambda^* w, \hat{P}f)$$

and

$$0 = [S_\lambda^* i_\lambda\hat{P}'f, w]_\lambda = (iS_\lambda w, \hat{P}'f)$$

so $\hat{P}'iS_\lambda w = \hat{P}iS^*w = 0$. Since $\langle\cdot,\cdot\rangle$ is local it follows that

$$\begin{aligned} 0 &= \langle S_\lambda^* w, S_\lambda w\rangle_\lambda \\ &= [S_\lambda^* w, w]_\lambda . \end{aligned}$$

Upon putting $v = S_\lambda^* w$ we have

$$0 = [v, T_\lambda^* v] = \langle v, v\rangle_\lambda = \| v \|_\lambda^2$$

and thus $w = T_\lambda^* v = 0$, $X_\lambda = \overline{M} \oplus \overline{N}$ and X_λ is decomposable. Thus (a) implies (b).

The final assertion of Lemma 5.7 is immediate from Lemma 5.2.

We summarize the results obtained above in the following.

THEOREM 5.1. *Let $\langle\cdot,\rangle$ be either symmetric or local and let $\lambda' > 0$. The following are equivalent.*

(a) ${}_aK_\lambda^* \subset Q \cup \{0\}$ *for* $0 \leqslant \lambda < \lambda'$,

(b) $K_{a,\lambda}^* \subset Q \cup \{0\}$ *for* $0 \leqslant \lambda < \lambda'$

(c) $\gamma_{\lambda'}$ *is ergodic and positive,*

(d) $\gamma_{\lambda'}$ *is positive and X is indecomposable.*

It only needs to be observed that, under the hypothesis of Theorem 5.1 the indecomposability of X is equivalent to that of X_λ for any $\lambda > 0$.

Remark. In the symmetric case condition (d) of Theorem 5.1 can be strengthened to read: X admits no proper orthogonal projection P such that both P and $I - P$ are positive.

COROLLARY 5.1. *Suppose that X is a Riesz space with $\langle u_+, u_- \rangle \leqslant 0$ for all $u \in X$ and X is indecomposable. Then ${}_aK_\lambda^*$, $K_{a,\lambda}^* \subset Q \cup \{0\}$ and γ_λ, γ_λ^* are strictly positive for all $\lambda \geqslant 0$.*

Remark. By virtue of Theorem 6.7, p. 270, [20], we gain no generality if in our initial assumptions in this section we replace the space $L^2(\Omega, \Sigma, \mu)$ by an arbitrary real Hilbert lattice and we lose no generality if we assume that Ω is a locally compact Hausdorff space and μ is a Radon measure on Ω. In the presence of the latter assumption we can, instead of assuming that the elements of X belong to $L^2(\Omega, \Sigma, \mu)$, assume merely that the elements of X are equivalence classes of locally μ-integrable functions and than for each compact set $F \subset \Omega$ there exists a constant $C(F)$ such that

$$\int_F |u|^2 \, d\mu \leqslant C(F) \| u \|, \qquad \text{all} \quad u \in X.$$

This is essentially the case considered in [9].

6. RIESZ SPACES

Let X be as in Section 5. Then our results imply that if X is a Riesz space and $\langle u_+, u_- \rangle \leqslant 0$ for all $u \in X$ then ${}_aK_\lambda^* \subset K$ for all $\lambda \geqslant 0$. The purpose of this section is to prove the converse. This result is equivalent in the symmetric case to a theorem of Deny, [9], and in the non-symmetric case to a theorem announced by Ito in [15]. We include a proof here for the sake of completeness and to illustrate the applicability of the techniques of this paper.

THEOREM 6.1. *Let the space X be as in Section 5 and assume that ${}_aK_\lambda^* \subset K$ for all $\lambda \geqslant 0$. Then X is a Riesz space and*

$$\langle u_+, u_- \rangle \leqslant 0, \qquad \textit{for all } u \in X \tag{6.1}$$

The proof depends on the following lemma.

LEMMA 6.1. (a) *The norm of S_λ as an operator in X is bounded independently of λ for $\lambda \geqslant 0$,* (b) *for $u \in X$, $\lambda > 0$,*

$$\langle u, S_\lambda u \rangle \geqslant \tfrac{1}{2} \| u \|^2, \tag{6.2}$$

(c) *the operator* $i(I - S_\lambda): X \to L^2$ *converges to zero in the operator norm and for* $u \in X$, $S_\lambda u$ *converges weakly to* u *as* $\lambda \to \infty$.

Proof. All three assertions follow directly from the identity

$$\langle u, S_\lambda u\rangle = \tfrac{1}{2}(\| u \|^2 + \| S_\lambda u \|^2) + \tfrac{1}{2} \| u - S_\lambda u \|^2 + \lambda \,|\, i(I - S_\lambda)\, u \,|^2, \quad (6.3)$$

which is obtained by addition of the identities

$$\begin{aligned}\langle u, S_\lambda u\rangle &= \| u \|^2 + \lambda(u, i(I - S_\lambda)\, u)\\ 0 &= \| S_\lambda u \|^2 - [u, S_\lambda u] - \lambda(iS_\lambda u, i(I - S_\lambda)\, u).\end{aligned}$$

These latter are obtained in turn as follows

$$\begin{aligned}\langle u, S_\lambda u\rangle &= \langle u, S_\lambda u\rangle_\lambda - \lambda(iu, iS_\lambda u)\\ &= \| u \|^2 - \lambda(iu, iS_\lambda u)\\ &= \| u \|^2 + \lambda(iu, i(I - S_\lambda)\, u)\\ 0 &= \langle S_\lambda u, S_\lambda u\rangle_\lambda - [S_\lambda u, u]_\lambda\\ &= \| S_\lambda u \|^2 - [u, S_\lambda u] - \lambda(iS_\lambda u, i(I - S_\lambda)\, u).\end{aligned}$$

Proof of Theorem 6.1. Let $u \in X$, then, since ${}_aK_\lambda^* \subset K$ for all $\lambda \geqslant 0$, it follows from Theorem 3.1 that there exists a $u_\lambda \in X$ for each $\lambda \geqslant 0$ such that

$$\| u_\lambda \|_\lambda \leqslant \| u \|_\lambda \quad (6.4)$$

$$S_\lambda(u_\lambda - u), \qquad u_\lambda + u \in K, \quad (6.5)$$

and

$$\langle u_\lambda + u, S_\lambda(u_\lambda - u)\rangle \leqslant 0 \quad (6.6)$$

or equivalently

$$\langle u_\lambda , S_\lambda u_\lambda\rangle \leqslant \langle u, S_\lambda(u - u_\lambda)\rangle + \langle u_\lambda , S_\lambda u\rangle. \quad (6.7)$$

From (6.2) and (6.7) there follows

$$\tfrac{1}{2} \| u_\lambda \|^2 \leqslant \| S_\lambda \| \,(\| u \|^2 + 2 \| u \| \, \| u_\lambda \|)$$

so that by Lemma 6.1, (a), $\| u_\lambda \|$ hence also $| u_\lambda |$ is bounded independently of λ. Let $\{\lambda_n\}$ be a sequence such that $\lim_{n=\infty} \lambda_n = \infty$ and $\{iu_{\lambda_n}\}$ is weakly convergent, say to w, in L^2. Then

$$iS_{\lambda_n} u_{\lambda_n} = iu_{\lambda_n} - i(I - S_{\lambda_n})\, u_{\lambda_n}$$

so by Lemma 6.3, $\{iS_{\lambda_n} u_{\lambda_n}\}$ also converges weakly to w. From (6.5) we then have

$$w + iu, \qquad w - iu \geqslant 0, \tag{6.8}$$

hence

$$| w | \geqslant | iu |.$$

On the other hand, by (6.4),

$$| u_\lambda |^2 \leqslant | iu |^2 + \lambda^{-1}(\| u \|^2 - \| u_\lambda \|^2),$$

so it follows that

$$| w | \leqslant \liminf_{\lambda_n \to \infty} | u_{\lambda_n} | \leqslant | iu | \leqslant | w |,$$

and thus

$$| w | = | iu | \tag{6.9}$$

and in fact $\{u_{\lambda_n}\}$ converges strongly to w in L^2. Moreover (6.8) and (6.9) imply that $w = | iu |$, and thus $w = | iu |$ is the unique limit point in L^2 of $\{u_\lambda\}$ as $\lambda \to \infty$. Furthermore as $\lambda \to \infty$, u_λ, $S_\lambda u_\lambda$ converge weakly in X to v, where $iv = w$. Since u was arbitrary it follows that X is a Riesz space.

To prove (6.1) we need to justify a passage to the limit in (6.6). To this end we write, using the identity (6.3)

$$\begin{aligned}\langle u_\lambda + u, S_\lambda(u_\lambda - u)\rangle &= \langle u_\lambda - u, S_\lambda(u_\lambda - u)\rangle + 2\langle u, S_\lambda(u_\lambda - u)\rangle \\ &\geqslant \tfrac{1}{2} \| u_\lambda - u \|^2 + \| S_\lambda(u_\lambda - u)\|^2 + 2\langle u, S_\lambda(u_\lambda - u)\rangle.\end{aligned}$$

By (6.6) and weak lower semi-continuity of the norm we have, with v as above

$$\| v - u \|^2 + 2\langle u, v - u\rangle \leqslant 0$$

or

$$\langle v + u, v - u\rangle \leqslant 0.$$

Since $v + u = 2u_+$ and $v - u = 2u_-$, this completes the proof.

For a more intrinsic characterization of the case in which X is a Riesz space and $\langle \cdot, \cdot \rangle$ is semi-local see Ancano, [3]. For the case in which X is a Riesz space with semi-local inner product $[\cdot, \cdot]$ representation theorems for $[\cdot, \cdot]$ are given in [2], [4].

7. Applications to Differential Equations

Let Ω' be a connected open set in R^N and let

$$\langle u, v\rangle = \int_{\Omega'} \sum_{|\alpha|, |\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\alpha u\, D^\beta v\, dx \tag{7.1}$$

be a Dirichlet form with locally integrable coefficients defined on Ω', $m \geqslant 1$; the notation used on the right in (7.1) is that of [1], [13]. Let Ω be a connected open subset of Ω' with $\bar{\Omega} \subset \Omega'$ and let $V = V(\Omega)$ denote the class of continuous functions u with compact support in Ω such that u has continuous derivatives of all orders α with $|\alpha| < m$ and $D^\alpha u$ is Lipschitz continuous whenever $|\alpha| = m - 1$. In order that we may apply the results of the previous sections, we make the following assumptions:

(1) for every semi-norm π on $H^{\text{loc}}_{m,1}(\Omega')$ there exists a constant C_π such that

$$(\pi(u))^2 \leqslant C_\pi \langle u, u \rangle \qquad \text{for all } u \in V,$$

(2) there exists a constant C such that

$$\langle u, v \rangle^2 \leqslant C \langle u, u \rangle \langle v, v \rangle, \qquad \text{all } u, v \in V, \tag{7.2}$$

and either:

(3) the quadratic form $\langle u, u \rangle$ is lower semi-continuous on V with respect to the $H^{\text{loc}}_{m,1}(\Omega')$ topology, or the stronger assumption:

(4) there exists a constant C' such that

$$\int_{\Omega'} \Big| \sum_{|\alpha|, |\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\alpha u\, D^\beta u \Big| \, dx \leqslant C' \langle u, u \rangle$$

for all $u \in V$.

Let X denote the completion of V in the norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. Then X is a Hilbert space and under the assumptions (1), (2), (3) above X can be identified with a linear manifold in $H^{\text{loc}}_{m,1}(\Omega')$, the functions in which vanish *a. e.* on Ω'/Ω. If (and only if) (4) holds then the inner product on X is given by (7.1) with $D^\alpha u$, $D^\beta v$ the distribution derivatives.

Since the elements of X can be identified with functions in $H^{\text{loc}}_{m,1}(\Omega)$ it follows from standard properties of the latter space that X is indecomposable (since Ω is connected) and X is a Riesz space if and only if $m = 1$; if (4) holds then $\langle \cdot, \cdot \rangle$ is local. In what follows we shall assume that (1), (2), (4) hold unless the form (7.1) is symmetric in which case we assume only (1), (2), (3).

Let μ be Lebesgue measure or some other measure on Ω which is locally equivalent to Lebesgue measure and let $L^2(\mu)$ be the space of real-valued functions that are μ-square-integrable over Ω with

$$| u | = \left(\int_\Omega u^2 \, d\mu \right)^{1/2}, \qquad u \in L^2(\mu).$$

Assume finally that there exists a constant C' such that

$$| u | \leqslant C' \| u \| \qquad \text{all } u \in X.$$

We consider the *weak Dirichlet problem* for $f \in L^2(\mu)$, $\lambda \geqslant 0$

$$\langle v, u\rangle + \lambda \int_\Omega uv \, d\mu = \int_\Omega vf \, d\mu, \qquad \text{all} \quad v \in X. \tag{7.3}$$

For X as defined above and $L^2 = L^2(\mu)$, we can apply the results of Sections 5 and 6 above, in particular γ_λ becomes in this case the solution operator for the problem (7.3), i.e. $u = \gamma_\lambda f$ is the solution of (7.3). Moreover, in view of the properties of X listed above we have the following.

THEOREM 7.1. *The operator γ_λ is positive for all $\lambda \geqslant 0$ if and only if $m = 1$. If $\lambda' \geqslant 0$ and $\gamma_{\lambda'}$ is positive then it is ergodic and γ_λ' is strictly positive for $0 \leqslant \lambda < \lambda'$.*

Remark. For the case in which the coefficients in (7.1) are sufficiently smooth (e.g. $a_{\alpha\beta} \in C^{|\beta|}(\Omega)$, all α, β) the first assertion of Theorem 7.1 has been proved by Calvert, [7].

THEOREM 7.2. *Let $\Omega = \bigcup_{n=1}^\infty \Omega_n$ where $\Omega_n \subset \Omega_{n+1}$ for $n = 1, 2,\ldots$ and each Ω_n is a connected open set. Let X_n be the completion of $V(\Omega_n)$ in X. If the solution operator for the Dirichlet problem*

$$\int_{\Omega_n} \sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\alpha u \, D^\beta v \, dx = \int_\Omega vf \, d\mu, \qquad \text{all} \quad v \in X_n, \tag{7.4}$$

is positive for $n = 1, 2,\ldots$, then so is that for

$$\int_{\Omega} \sum_{|\alpha|,|\beta| \leqslant m} a_{\alpha\beta}(x)\, D^\alpha u \, D^\beta v \, dx = \int_\Omega vf \, d\mu, \qquad \text{all} \quad v \in X. \tag{7.5}$$

Proof. We have $X_n \subset X_{n+1}$, $n = 1, 2,\ldots$ and $\bigcup_{n=1}^\infty X_n$ is dense in X. It was observed in [6] that condition (d) of Corollary 3.1 need only be checked on a dense subset of X. The situation is similar with regard to Theorem 3.1. Indeed condition (c) of that theorem implies the following: (c') for every $x \in X$ there exists a $w(= Sv)$ with $\| w \| \leqslant \| x \|$, $w \in K$ and $\langle x, w\rangle \leqslant -(\operatorname{dist}(x, K))^2$ and since this condition implies condition (g) of Theorem 3.1 it is equivalent to (c). It is clear that if (c') holds on a dense subset of X then it holds on all of X. Thus if the solution operator for (7.4) is positive for each n then by Lemma 5.3 and Theorem 3.1, condition (c') is satisfied for each X_n and consequently for X. By another application of Lemma 5.3 and Theorem 3.1 it follows that the solution operator for (7.4) is positive.

8. The Hadamard Conjecture

The Hadamard conjecture, see [11], is to the effect that the solution u of the problem

$$\Delta^2 u = f \quad \text{in} \quad \Omega\,(\subset R^2), \qquad u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{8.1}$$

is non-negative when $f \in L^2(\Omega)$ is non-negative. Numerous counter-examples are known, the earliest is due to Duffin, [10] another interesting counter-example is that of Garabedian, [14]. We shall show in this section that any convex bounded region Ω can be altered to provide a counter-example to the Hadamard conjecture by "stretching" and any region can be so altered by the excision of a sufficiently small neighborhood of an arbitrary interior point.

Let Ω be a region in R^2 and assume (H): Ω is contained in a region Ω' whose harmonic Green's function $G(z, \zeta)$ ($z = x + iy$, $\zeta = \xi + i\eta$, x, y, ξ, η real satisfies

$$\sup_{z\in\Omega} \int_{\Omega'} | G(z, \zeta)|^2 \, d\xi \, d\eta < \infty,$$

(this includes in particular an infinite straight strip as well as any bounded region Ω). A region satisfying (H) will be called *admissible.*

For an admissible region Ω the space $H_0^2(\Omega)$ can and in what follows will be normed by

$$\| u \| = \left(\int_\Omega | \Delta u |^2 \, dx \, dy\right)^{1/2}, \tag{8.2}$$

and the problem (8.1) is meaningful, at least in the weak sense for any $f \in L^2(\Omega)$.

From Theorem 7.2 we immediately obtain the following.

THEOREM 8.1. *Let Ω be an admissible region and let $\{\Omega_n\}$ be an increasing sequence of connected open subsets of Ω with $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. If the Hadamard conjecture is false for Ω then it is false for all but a finite number of the Ω_n.*

Let Ω be a region in the plane such that: (a) Ω admits two parallel tangent lines L_1 and L_2 and Ω is contained in the infinite strip whose boundary is $L_1 \cup L_2$, (b) there is a line segment C joining L_1 and L_2 and whose interior lies in Ω; in particular any bounded convex region satisfies (a) and (b). For such a region Ω let L_1, L_2 and C be fixed and for $a > 0$ let T_a denote the unique orientation preserving affine transformation which leaves L_1, L_2 and C invariant and is such that for $z, \zeta \in L_1$

$$| T_a z - T_a \zeta | = a \,| z - \zeta |.$$

It is clear then that $\bigcup_{a>0} T_a\Omega$ is the infinite strip with boundary $L_1 \cup L_2$. Thus, by appealing to Duffin's result concerning the infinite strip we have the following.

COROLLARY 8.1. *Let Ω, T_a be as above, then for all sufficiently large a, $T_a\Omega$ violates the Hadamard conjecture.*

In particular a long thin rectangle or a long thin ellipse will violate the conjecture. Garabedian has shown, [14], that in fact the conjecture is violated by an ellipse whose major axis has no more than twice the length of the minor axis.

Another well-known counter-example is that of the punctured disk. In fact, it has been shown, [12], [21] that the Szëgo conjecture, see [11], is false for such a region and *a fortiori* the Hadamard conjecture is also.

We shall give an extremely simple proof that any admissible region whose boundary contains an isolated point violates the Hadamard conjecture. It will then follow readily from Theorem 8.1 that if Ω' is an admissible region and z an interior point of Ω' then for a sufficiently small neighborhood N of z the region $\Omega = \Omega' \backslash N$ will violate the Hadamard conjecture.

We require the following.

LEMMA 8.1. *Let* 0 *be an isolated point of the boundary of the plane region* Ω,

(a) *if* $u \in H_0^2(\Omega)$ *then*

$$\lim_{r\to 0} \frac{1}{r} \int_{|z|=r} u(z) \frac{|dz|}{|z|} = 0 \tag{8.3}$$

(b) *if* $u \in C_0^\infty(\Omega \cup \{0\})$ *and* $u(0) = 0$ *then* $u \in H_0^2(\Omega)$.

Remark. Lemma 8.1 implies that $H_0^2(\Omega)$ is of co-dimension 1 in $H_0^2(\Omega \cup \{0\})$. Since these spaces are reproducing kernel spaces this fact enables one to compute the reproducing kernel for $H_0^2(\Omega)$ normed by (8.2), i.e. the Green's function (8.1), when that for $H_0^2(\Omega \cup \{0\})$ is known.

We next state the following consequence of Lemma 8.1, the proof of the Lemma will then be indicated.

PROPOSITION 8.1. *Let Ω be an admissible region such that* 0 *is an isolated point of $\partial\Omega$. Then the positive cone K in $H_0^2(\Omega)$ is not generating, i.e. $H_0^2(\Omega) \neq K - K$ and consequently Ω violates the Hadamard conjecture.*

Proof. By Lemma 8.1 there exists a function $w \in H_0^2(\Omega)$ such that $w(x + iy) = x$ for $x^2 + y^2$ sufficiently small. If $w = u - v$ where $u, v \geqslant 0$ on Ω then u will fail to satisfy (8.3), and thus $u \notin H_0^2(\Omega)$ and K is not generating. It follows from Corollary 3.1 and Lemma 5.3 that Ω violates the Hadamard conjecture.

Proof of Lemma 8.1. The proof is by standard arguments so only an indication will be given. For (a) we first assume that u depends only on $r = (x^2 + y^2)^{1/2}$ near 0, then for such a u and small r,

$$u(r) = \int_0^r \ln \frac{r}{\rho} \left(\frac{\partial}{\partial \rho} \rho \frac{\partial u}{\partial \rho}\right) d\rho$$

so by the Schwarz inequality

$$| u(r)|^2 \leqslant \int_0^r \rho \left(\ln \frac{r}{\rho}\right)^2 d\rho \int_0^r \left(\frac{\partial}{\partial \rho} \rho \frac{\partial u}{\partial \rho}\right)^2 \frac{d\rho}{\rho},$$

and the assertion follows. For general $u \in H_0^2(\Omega)$ we can, after using a partition of unity, assume u vanishes identically outside a small neighborhood of 0 and then consider

$$v(r) = \frac{1}{2\pi} \int_{|z|=r} u(z) \frac{| dz |}{| z |} \in H_0^2(\Omega).$$

If u is as in (b) and in addition

$$| u(z)|, \ | z | \ | \operatorname{grad} u(z)|, \ | z |^2 \ | \Delta u(z)| = O(| z |^{1+\epsilon})$$

as $| z | \to 0$ for some $\epsilon > 0$ then it is easily verified that $u \in H_0^2(\Omega)$. In view of Taylor's Theorem it remains only to consider functions u, $v \in C_0^\infty(\Omega \cup \{0\})$ such that on some neighborhood of 0

$$u(x + iy) \equiv x, \qquad v(x + iy) \equiv y.$$

Assume that two such functions are given and let, for $\epsilon > 0$,

$$u_\epsilon(z) = | z |^\epsilon u(z), \qquad v_\epsilon(z) = | z |^\epsilon v(z).$$

then u_ϵ, $v_\epsilon \in H_0^2(\Omega)$ for all $\epsilon > 0$ and an elementary computation shows that

$$\lim_{\epsilon \to 0} \| u_\epsilon - u \| = \lim_{\epsilon \to 0} \| v_\epsilon - v \| = 0,$$

where $\| \cdots \|$ is the norm defined by (8.2). It follows that u, $v \in H_0^2(\Omega)$.

9. Applications to Semi-Groups

Let $S(t)$ be a strongly continuous semi-group of contractive linear operators on (real) $L^2 = L^2(\Omega, \Sigma, \mu)$ with generator $-A$. We can assume without loss of generality that there exists a constant $\mu > 0$ such that

$$(Au, u) \geqslant \mu \, | u |^2, \qquad u \in L^2, \tag{9.1}$$

and we shall also assume that the symmetric part of A dominates the skew symmetric part, i.e., that there exists a constant C such that

$$(Au, v) \leqslant C(Au, u)(Av, v), \qquad u, v \in \mathscr{D}(A); \tag{9.2}$$

$\mathscr{D}(A)$ denotes the domain of A.

We first give a refinement for this particular case of a result of Phillips, [19], characterizing the generator of a positivity preserving semi-group on a Banach lattice. Secondly, we generalize a result proved by Simon, [22], for the self-adjoint case and characterizing strict positivity of the semi-group $S(t)$. We note that the assumption that the symmetric part of A is dominant rules out in par ticular a semi-group on L^2 which is induced by a semi-group of measure preserving transformations on Ω.

Let X denote the completion of $\mathscr{D}(A)$ with respect to the inner product.

$$[u, v] = \tfrac{1}{2}((Au, v) + (u, Av)), \tag{9.3}$$

and denote by i the continuous extension to X of the inclusion map $\mathscr{D}(A) \subset L^2$; we have, by (9.1),

$$| iu | \leqslant \mu^{-1} \| u \|, \qquad u \in X; \tag{9.4}$$

the inner product on X is denoted by $[\cdot, \cdot]$ and the norm by $\| \cdot \|$, i.e. $\| u \|^2 = [u, u]$, $u \in X$. By (9.2) the bilinear form (Au, v) on $\mathscr{D}(A)$ extends to a continuous bilinear form $\langle \cdot, \cdot \rangle$ on X and

$$\langle u, v \rangle = (Au, iv), \qquad \text{for} \quad u \in \mathscr{D}(A), \quad v \in X, \tag{9.5}$$

while by (9.3)

$$[u, v] = \tfrac{1}{2}(\langle u, v \rangle + \langle v, u \rangle), \qquad u, v \in X. \tag{9.6}$$

It follows from (9.5) and (9.6) that i is injective and thus the elements of X can be identified with (equivalence classes of) functions in L^2. Let $w \in \mathscr{D}(A^*)$, i.e. let there exist a constant C' such that

$$|(Au, w)| \leqslant C' | u |, \qquad \text{all } u \in \mathscr{D}(A)$$

then in view of (9.4) the linear functional $(A(\cdot), w)$ extends to a continuous linear functional on X and thus there exists a $v \in X$ such that

$$\langle u, v \rangle = (Au, w), \qquad u \in \mathscr{D}(A).$$

Since i is injective it follows from (9.5) that $v = w$ and thus, since $w \in \mathscr{D}(A^*)$ was arbitrary, $\mathscr{D}(A^*) \subset X$. Finally we note that since i is injective there exists a unique self-adjoint positive definite operator B on L^2 such that

$$[u, v] = (B(iu), iv), \qquad u, v \in X,$$

and X coincides with $\mathscr{D}(^{1/2})$; B is the *symmetric part* of A.

LEMMA 9.1. *Let the generator* $-A$ *of the semi-group* $S(t)$ *satisfy* (9.1) *and* (9.2), *then* $S(t)$ *is analytic.*

Proof. We first extend A, B to the space L_c^2 of complex-valued μ-square-integrable functions and then note that if λ is a complex number with $\operatorname{Re}\lambda > 0$ then

$$|(Au, u) + \lambda(u, u)| \geqslant |\lambda| \, \| u \|^2, \qquad u \in \mathscr{D}(A) \tag{9.7}$$

and

$$|(B^{1/2}u, B^{1/2}u) + \lambda(u, u)| \geqslant |\lambda| \, \| u \|^2, \qquad u \in \mathscr{D}(B^{1/2}) \tag{9.8}$$

If for a fixed λ the range of $A + \lambda I$ is not dense in L^2 then there exists $w \in L^2$ such that

$$(Au, w) + \lambda(u, w) = 0, \qquad \text{all} \quad u \in \mathscr{D}(A).$$

This implies however that $w \in \mathscr{D}(A^*) \subset \mathscr{D}(B^{1/2})$ and thus by (9.8) $w = 0$. It follows that the right half-plane is contained in the resolvant set of A and that

$$\|(\lambda I + A)^{-1}\| \leqslant \frac{1}{|\lambda|}, \qquad \text{for} \quad \operatorname{Re}\lambda > 0.$$

It follows from standard results (e.g., Theorem 2.1, p. 101, [13]) that $S(t)$ is analytic.

By applying Theorem 6.1 we obtain the following characterization of those semi-groups $S(t)$ which satisfy the above conditions and are positivity preserving, cf. Phillips, [19].

THEOREM 9.1. *The semi-group* $S(t)$ *is positivity preserving if and only if* X *is a Riesz space and*

$$\langle u_+ , u_- \rangle \leqslant 0, \qquad u \in X.$$

Proof. As is well known, $S(t)$ is positivity preserving if and only if the resolvant $(\lambda I + A)^{-1}$ is positivity preserving for all real positive λ. This is an immediate consequence of the resolvant formula and the Hille approximation formula for $S(t)$. The assertion is thus an immediate consequence of Theorem 6.1.

Let Ω be a locally compact Hausdorff space and let μ be a Radon measure on Ω. Suppose also that $\mu(\Omega) < \infty$ and $1 \in \mathscr{D}(A)$. Then Theorem 9.1 together with the main result of [2] implies the following representation for the symmetric part of A when $S(t)$ is positivity preserving, namely

$$(Bu, v) = \int_\Omega uv \, d\sigma + \int_{(\Omega\times\Omega)\setminus\Delta} (u(x) - u(y))(v(x) - v(y)) \, d\nu + N(u, v)$$

where $\Delta = \{(x, x) : x \in \Omega\}$ σ and ν are Radon measures on Ω and $(\Omega \times \Omega)\setminus\Delta$ respectively and N is a non-negative definite local bilinear form such that $N(u, 1) = 0$ for all $u \in \mathscr{D}(B)$.

LEMMA 9.2. *If the semi-group $S(t)$ is positivity preserving then for any $f \in L^2$ with $f > 0$*

$$(S(t)f, f) > 0, \qquad t > 0.$$

Proof. Let $\{t_n\}$ be a sequence of positive numbers converging to 0. Then for $\epsilon > 0$,

$$\{x \in \Omega : f(x) > \epsilon\} \subseteq \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x \in \Omega : f(x), (S^*(t_k)f)(x) \geqslant \epsilon\}$$

thus, for ϵ small enough there exists an n_0 such that

$$E = \bigcap_{k=n_0}^{\infty} \{x \in \Omega : f(x), (S^*(t_k)f)(x) \geqslant \epsilon\}$$

has positive measure. Let ϵ, n_0, E be so determined. For any $t > 0$ and $k \geqslant n_0$ we have

$$(S(t)f, f) = (S(t - t_k)f, S^*(t_k)f)$$

provided $t_k < t$ and thus

$$(S(t)f, f) \geqslant \epsilon \int_E S(t - t_k)f \, d\mu. \tag{9.4}$$

The integral of $S(t)f$ over E is analytic in t and not identically zero, so t_k can be chosen so that the integral on the right in (9.4) does not vanish. This completes the proof.

THEOREM 9.2. *Let $S(t)$ be positivity preserving, then the following are equivalent*:

(a) $(\lambda I + A)^{-1}$ *is strictly positive for all* $\lambda > 0$,

(b) X *is indecomposable*,

(c) $S(t)$ *is ergodic*,

(d) $S(t)$ *is strictly positive for* $t > 0$.

Proof. The equivalence of (a) and (b) follows from the results of section 5. The equivalence of (a) and (c) follows from the resolvant formula and (c) is clearly implied by (d). By virtue of Lemma 9.1 the proof given by Simon, [22] for the implication (c) implies (d) in the symmetric case can be extended to the more general case considered here. This goes as follows, if $f, g \in L^2$, $f, g > 0$ then by ergodicity and analyticity $(S(t)f, g) > 0$ except for a discrete set of positive values of t. Let $t_1 > 0$ and let $(S(t_1)f, g) > 0$, then there exists a $w \in L^2$ such that

$$0 < w < \inf\{S(t_1)f, g\}.$$

Thus for $t > t_1$

$$(S(t)f, g) \geqslant (S(t - t_1)w, w) > 0.$$

Since t_1 can be taken to be arbitrarily small it follows that $S(t)$ is strictly positive for $t > 0$.

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